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The vortex: complex Hopf bundle and Morse theory

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Abstract. We interpret the vortex solution of Nielsen–Olesen as a complex vector bundle associated to the second Hopf sphere bundle (analogously to consider the kink of the first Hopf bundle); the peculiarity of the soliton behaviour of the two-dimensional vortex stems from the non-trivial character of the fibration; the electromagnetic and scalar (Higgs) field are the connection and the section in this bundle respectively.

Properties of this mathematical construction have their natural physical translation; for example the complex structure of the sphere S^2 leads to a closed (Kähler) two-form, which has physical implications, and the fact that the vortex can be considered as the square root of the tangent bundle to the sphere implies a spinor nature for the vortex.

A Morse theory of critical points suggests some Atiyah–Singer type of theorems, which have bearing on the stability of the multi-vortex solutions.

We finish by a geometrical interpretation of the fractional charges found recently.

1. Introduction

Peculiar solutions to classical field equations, called generically solitons, have been intensively studied in recent years in particle physics (Coleman 1977); they arise most naturally in some theories exhibiting spontaneous symmetry breakdown. More solid ground for broken symmetries is provided in the framework of fibre bundles (rather than a shrewd choice of the potential) since in the fibre bundle the topological charge is another name for the characteristic number of the bundle and the sections and connections become the matter (Higgs) and gauge fields, respectively; and the physical gauge group is naturally the structure group of the bundle. The first example was neatly worked out by Wu and Yang (1975).

In this paper, which follows a short note (Boya and Mateos 1980a, b), we concentrate on the mathematical structure of a particular two-dimensional soliton (the vortex of Nielsen and Olesen (1973)) by describing the scalar field in it as a definite function (section) in a non-trivial vector bundle over the (compactified) spatial part of a space-time: the ultimate reason for solitons seems to be the twisted 'wedding' of the internal symmetry group G (the gauge or structural group) with the (topological non-trivial) manifold representing space-time.

This paper is organised as follows: in §§2 and 3 the bundle nature is explained and characterised as the associated one-dimensional complex vector bundle of the second Hopf sphere bundle with exact sequence $\beta : S^1 \rightarrow S^3 \rightarrow S^2$. We show how the boundary conditions give rise to the non-trivial fibration, how the localisation of the source of the magnetic field leads to an asymptotically flat connection, etc. The complex structure, spinor character and characteristic class give mathematically sound grounds for the stability conditions, statistics, and topological charge, respectively. In

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particular, the quantisation of the total flux is just the integral realisation of the (first) Chern class of the bundle (Gauss–Bonnet theorem).

Sections 4 and 5 are devoted to stability considerations of the vortex equations by means of the Morse theory (e.g. Milnor 1963); in the limit $\gamma = e^2/\lambda = 1$ the saturation of the Bogomolny bound (Bogomolny 1976) leads to first-order equations related to the Kähler structure (Perelomov 1978), subsequent to the complex structure mentioned above. Atiyah–Singer type theorems (Palais 1965), set up in § 6, strongly suggest stability of the multivortex solution. Finally in § 7 we establish a relationship between the nature of the spin bundle associated to the vortex and the recently found fractional charges for fermions (Jackiw and Schrieffer 1981).

Extension of these considerations for monopoles and instantons are left for a later publication.

2. The bundle framework

In this section we are to specify the bundle structure appropriate to describe the vortex of Nielsen and Olesen (1973). We remember that the vortex is a static solution of a system of field equations in $(2+1)$ space-time describing scalar electrodynamics; the vector potential $A = A_\mu$ becomes a pure gauge at spatial infinity and the matter field Φ takes there a non-zero value (because of spontaneous symmetry breakdown).

This suggests the following construction of a bundle with connection: as base space, we take it to be S^2, the bi-dimensional sphere, as a prolongation of the ordinary space part R^2 with the one-point compactification, $S^2 = R^2$ plus $\{\infty\}$, with $\{\infty\}$ the set of a point at infinity. We can even think of a (conformally invariant) stereographic projection $S^2 \leftrightarrow R^2 \cup \{\infty\}$ and write the field equations in S^2, but we shall not bother to do that, because the system is not conformally invariant and the form of the equations would change with the common ones in R^2. Of course, the topological structure which emerges from defining the fields in R^2 with the vortex boundary conditions (which maintain finite energy) is the same as the topology obtained by defining fields on S^2.

Over S^2 we now erect the so-called complex Hopf bundle (see e.g. Steenrod 1951), which we designate $\beta: P(S^2, S^1)$. The bundle β is characterised in the following way: given a covering of S^2 by open sets $\{U_+, U_-\}$, excluding respectively the south and north poles, with an overlap homeomorph to the open cylindrical strip, $U_+ \cap U_- \approx S^1 \times (0, 1)$, the bundle gets fixed once a transition function $U_+ \cap U_- \to \mathrm{G}$, where G is the structure group homeomorph to the fibre, is given. The group G is here $\mathrm{U}(1) \approx S^1$, the gauge group for electromagnetism (Weyl 1923). Because the strip $(0, 1)$ is contractible, in order to fix (the class of equivalence of) the bundle it is enough to give a homotopy class of maps $S^1 \to S^1$. *Our bundle will be 'the class one'* ('windunghzal' = winding number = 1) in the set of maps $S^1 \to S^1$, which is precisely the mathematical definition of the second (complex) Hopf sphere bundle. If $S_\pm: U_\pm \to \pi_p^{-1}(U_\pm) \approx U_\pm \times \mathrm{G}$ are the trivialising sections, the transition function, $S_+ = g_{+-}S_-$, $g_{+-}: S^1 \times (0, 1) \to \mathrm{G}$, is $g_{+-}(\theta, x) = \mathrm{e}^{\mathrm{i}\theta}$. The total space of the bundle is S^3. The Hopf construction is: take all quaternions ($\approx R^4$) of norm one ($\approx S^3$), modulus unit complex numbers ($\approx S^1$); the quotient is a sphere S^2, $\beta: S^1 \to S^3 \to S^2$.

The *associated* vector bundle is obtained through a linear representation $D: \mathrm{G} \hookrightarrow$ vector space; here let us take $D: \mathrm{U}(1) \to R^2 = C$, $D(g)Z = \mathrm{e}^{\mathrm{i}\theta}Z$, $Z \in C$. Let $E = E(\xi)$ be the total space of this complex vector bundle, $E(\xi) = S^3 \times C \bmod \mathrm{G}(= S^1)$

and $\pi_E: E \to S^2$ the projection. *Sections* in this bundle $\psi: S^2 \to E(\xi)$, $\pi_E \circ \psi = \mathrm{id}_{S^2}$ are the objects of primary physical importance; in coordinates, ψ appears as a couple of functions $\psi_\pm: U_\pm \to C$ that are gauge connecting in the overlap:

$$\psi_+(u) = D(g_{+-})\psi_-(u), \qquad u \in U_+ \cap U_- \tag{2.1}$$

which can be seen as the gauge transformation of the scalar (matter) complex fields. As the bundle is non-trivial there are no sections without zeros. The sections will be identified with the Higgs field, which have to be zero somewhere.

We also recall that a *connection* in a bundle $P(B, \mathrm{G})$ is given by a G-Lie-algebra-valued one-form on P, say Θ. For $\mathrm{G} = \mathrm{U}(1)$, Lie-algebra = R, we have usual one-forms, i.e. R-valued: if $S_\pm$ are the trivialising sections $S^2 = \text{base} \to S^3 = \text{total space of}$ the principal bundle, as before, the pull-back $A_\pm = S_\pm^* \Theta$ are one-forms on the base manifold, identical with the ordinary physical electromagnetic vector potential, $A_\mu{}^\pm$, $\mu = 1, 2$: because the bundle is non-trivial, we need to specify A_μ independently in two patches, and for points in the overlap, $z \in U_+ \cap U_-$ we have (Wu and Yang 1975)

$$A_+(z) = A_-(z) + g_{+-}^{-1}(z)\, \mathrm{d}g_{+-}(z) \tag{2.2}$$

which is the usual gauge transformation of the vector potential.

The curvature $\Omega = D\Theta$ is an (R-valued) two-form on $P = S^3$, which gives the *unique* field strength $F = S^*\Omega = \mathrm{d}A$: the shift in the overlap is zero for F because $\mathrm{d}d = 0$: the electromagnetic field strength is a well defined two-form over the whole base manifold S^2, and it is (because U(1) is abelian) gauge invariant. As is well known, F as curvature measures to what extent the *covariant derivative* $D\Phi = (\partial - \mathrm{i}A)\Phi$ of the sections in the associated vector bundle is path dependent.

Our bundle β possesses also a *complex structure*. We remember (Milnor 1974) that a complex structure J on a real $2n$-dimensional vector bundle ξ is a continuous function $J: E(\xi) \to E(\xi)$ linear into each fibre and anti-involutory: $J^2 = -id$; equivalently, the structure group, in general $\mathrm{GL}(2n; R)$, reduces to $\mathrm{GL}(n; C)$. Our bundle β is 'already' reduced, because $\mathrm{U}(1) = \mathrm{SO}(2) \subset \mathrm{O}(2) \subset \mathrm{GL}(2, R)$ and in fact the further reduction $\mathrm{GL}(1, C) = C^* \to \mathrm{U}(1)$ implies also a *Hermitian metric h*, that is, a Hermitian form on each fibre.

The construction of h goes as follows. In $S^2 = 1 - D$ complex projective space = CP^1, with points $z \in S^2$, we have the standard metric $g = \mathrm{d}s^2 = a^2\,(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)$, which by complexification and stereographic projection gives

$$g(z) = \frac{\mathrm{d}\bar{z}\,\mathrm{d}z}{(1 + |z|^2)^2}; \tag{2.3}$$

this induces the Hermitian metric in $E(\xi)$ as

$$h(v) = \frac{\mathrm{d}\bar{v}\,\mathrm{d}v}{(1 + |v|)^2} \qquad \text{for } v \in \pi_E^{-1}(z).$$

The antisymmetric part of h gives us a symplectic structure, that is a two-form (called the *Kähler form* of the Hermitian metric),

$$W = \mathrm{i}\,\frac{\mathrm{d}\bar{v} \wedge \mathrm{d}v}{(1 + |v|^2)^2} = \mathrm{i}\,\frac{\mathrm{d}\bar{\phi} \wedge \mathrm{d}\phi}{(1 + \bar{\phi}\phi)^2} = \frac{\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b}{(1 + \phi^a\phi^a)^2}; \qquad a, b = 1, 2 \tag{2.4}$$

with $\pi_E^{-1}(z) = V(z) = \phi(z) = \phi^1(z) + \mathrm{i}\phi^2(z)$ for some section (or Higgs or matter field)

$\phi : S^2 \to E(\xi)$. This Kähler form W is here closed, $\mathrm{d}w = 0$, or

$$\frac{\partial}{\partial \phi^a} \cdot \frac{1}{(1+\phi^a\phi^a)^2} = \frac{\partial}{\partial \phi^b} \cdot \frac{1}{(1+\phi^a\phi^a)^2} \tag{2.5}$$

and $E(\xi)$ is a Kähler manifold.

3. Characteristic class

Characteristic classes are topological invariants of vector bundles, which can be obtained in various ways and are fundamental in distinguishing inequivalent bundles (Milnor 1974). For *complex* n-dimensional bundles we have k ($k = 1 \ldots n$) Chern classes, so here we just have $C_1 = C_1(\beta)$, the first; it can be defined axiomatically $C_1(\beta) = 1$ (Hirzebruch 1966) or computed from the Čech cohomology characterisation of the bundle. From the physical point of view it is a topological quantum number.

The set of equivalence classes of principal G-bundles over a manifold B is a cohomology set, $\check{H}^1(B, \mathrm{G})$ ($\check{}$ for Cech). For $B = S^n$, however, it is a standard result that $\check{H}^1(S^n, \mathrm{G}) = \pi_{n-1}(\mathrm{G})$, where $\pi_k(X)$ is the kth homotopy group of space X (this follows easily from the discussion of § 2 on maps $(0, 1) \times S^1 \to \mathrm{G}$). Here of course $\pi_1(\mathrm{U}(1)) = \pi_1(S^1) = Z$, and the n realised by a particular ($\mathrm{G} = \mathrm{U}(1)$, $B = S^2$) bundle is its Chern class; here again $C_1(\beta) = 1$ as follows from the definition of the second Hopf sphere bundle. Another proof comes from the identity $\check{H}^1(B, \mathrm{G} = S^1) = H^2(B, \mathrm{G} = Z)$, which comes from $R/Z = S^1$ and contractibility of R; as the sphere S^2 is connected ($\pi_0 = 0$) and simply connected ($\pi_1 = 0$), it follows easily that $H^2(S^2) = \pi_2(S^2) = Z$.

There is, moreover, a differential–integral characterisation of Chern classes, which has considerable physical significance. The (first) Chern class of G-bundle over a two-dimensional manifold, V_2, is given by the integral of the curvature two-form F of any connection in the bundle, namely

$$C_1(\xi) = \frac{1}{2\pi} \int_{V_2} F \tag{3.1}$$

which is the content of the *Gauss–Bonnet theorem* (Milnor 1974). Physically equation (3.1) characterises the quantisation of the magnetic flux: in our case $F = B = F_{12}$; $V_2 = S^2$, and $C_1(\xi) = (2\pi)^{-1} \int_{S^2} F = 1$; in this case the first Chern class is the last, and (3.1) identifies also the *Euler class* of the bundle, $\chi(\beta) = C_1(\beta) = 1$ (Milnor 1974).

There are two important mathematical developments from (3.1) with physical significance; the first is the dual statement: the *Poincaré–Hopf* theorem asserts (Chern 1956) that the Euler number χ can be obtained by counting the 'windungzahl' of zeros of sections; it says explicitly

$$C_1(\xi) = \sum_{p_i \text{ zero}} V(p_i) \tag{3.2}$$

where p_i ($i = 1, 2 \ldots$) are the zeros of the section in the associate bundle with windungzahl $V = V(p_i) \in Z$. The physical implication is that scalars (Higgs) in the vortex have a zero at the origin and that any n-multivortex solution should have n zeros at least.

The other statement comes from obtaining (3.1) by a boundary integral. The *Stokes* theorem permits a physicist's interpretation (and calculation) of (3.1). F is closed but not exact (as two-form), hence it has a de Rham period; we put $F \sim \mathrm{d}A$

by describing S^2 by $\{U_+, U_-\}$, leaving U_- 'at infinity', and then the calculation of the total flux gives

$$\frac{1}{2\pi}\int_{S^2} F = \frac{1}{2\pi}\int_{U_+\cup U_-} F \simeq \frac{1}{2\pi}\int_{\Sigma=U^+} F = \frac{1}{2\pi}\int_{\Sigma} \mathrm{d}A = \frac{1}{2\pi}\oint_{\partial\Sigma} A = 1 \tag{3.3}$$

for the $N-0$ vortex.

This computation of a topological charge by the integration over a boundary occurs time and again in physics (for example in general relativity).

After characterising the first Chern class of the bundle β in all these ways, we have that in the set of bundles $P(B, \mathrm{G})$ with $B = S^2$ and $\mathrm{G} = \mathrm{U}(1)$, which is $\check{H}^1(S^2, S^1) = H^2(S^2, Z) = Z$, the $n = 0$ is obviously the trivial bundle $S^2 \times S^1$, the $n = 1$ is our vortex and the $n = -1$ the 'antivortex' (easy to write down; notice that there is no 'antikink' (Boya and Mateos 1980b)). It is interesting to note that the $n = 2$ class is precisely the tangent bundle of the sphere S^2 (if only because $\chi(S^2) = (b_0 - b_1 + b_2)$ (sphere) $= +1 - 0 + 1$, with b_i the Betti numbers given (for example) by de Rham cohomology); as this bundle is therefore the square of the vortex (in a precise homological sense), *the vortex realises a sort of square root* of the tangent bundle to the sphere; it is more 'peculiar' than the sphere, as $\mathrm{i} = \sqrt{-1}$ is an 'enlargement' of real numbers; it is so reminiscent of Dirac's 1928 square root of quadratic forms, that we are able to identify the vortex bundle as a *spinor bundle* over the tangent bundle of the sphere (compare the identical situation for the sphere S^1 in the kink (Boya and Mateos 1980b).

The tangent bundle τ of a Riemann manifold $V = V_n$ has as principal bundle $\tau: \mathrm{O}(n) \to B \to V_n$; the spin map $Z_2 \to \mathrm{Spin}\,(n) \to \mathrm{SO}(n)$ 'lifts' τ to the *spinor bundle* $\tilde{\tau}$: $\mathrm{Spin}\,(n) \to B \to V_n$ if V_n is orientable and if there is no topological obstruction (anulation of the second Stiefel–Witney class of V_n). In our case $V_n = S^2$, which is Riemann, orientable, and with zero Stiefel–Witney classes; the tangent bundle τ: $\mathrm{SO}(2) \to B = \mathrm{SO}(3) \to S^2$ lifts to a covering bundle

$$\begin{array}{rlll}
 & \quad\alpha & & \\
Z_2 = & S^0 = & S^0 & \\
 & \downarrow & \downarrow & \\
\beta: & S^1 \to & S^3 = \mathrm{SU}(2) \to & S^2 = CP^1 \\
 & \downarrow & \downarrow & \quad \| \\
\tau = \beta^2: & S^1 \to & \mathrm{SO}(3) & \to S^2
\end{array}$$

because of the double covering $Z_2 \to \mathrm{U}(1) = S^1 \to \mathrm{SO}(2) = S^1$.

A spinor field is generally a section in the natural complex vector bundle associated with the spinor bundle; it follows that the scalar (Higgs) fields are spinor fields in that sense: therefore a natural *spinorial character* arises for the vortex (physically one should expect from this that two classical vortices anticommute; in fact this is so, and the proof is similar to the kink case; see later, § 4). One should remember also that a spinor does not return to its initial value except after *two* full rotations.

Notice also the first (or α) Hopf bundle appearing as the first vertical row. Thus the diagram reinforces the conception of the kink as a 'meron' vortex and the strong relation between merons and fermions in two space (Callan *et al* 1977).

4. The vortex field solution

We now study the vortex equation more closely; in the set of sections $\Gamma(\xi)$ of the complex vector bundle ξ associated to the Hopf bundle β we define the energy (or

Euclidean action) functional

$$S[\phi]=v^2\int\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2}\left\{|D\phi|^2+|D^+\phi|^2+2|F|^2+\frac{\gamma}{2}(|\phi|^2-1)^2\right\}\tag{4.1}$$

where $\phi: S^2\to E(\xi)$ is a section, i.e. $\phi(z)$ is a complex number, $D\phi=(\partial-\mathrm{i}A)\phi$ the usual covariant derivative for a connection (potential vector) $A=(A_1-\mathrm{i}A_2)/\sqrt{2}$, $D^+\phi=(\bar{\partial}-\mathrm{i}\bar{A})\phi$, $v^2=2m^2/\lambda$, $\gamma=\lambda/e^2$, and m, e, λ are the usual mass, gauge coupling and quartic coupling of the model. We really work in R^2 but consider the behaviour at infinity of the vortex type, namely

$$\begin{aligned}&|\phi|\to 1+\theta(|z|^{-1}) \qquad \arg\phi\to\theta+\mathrm{O}(|z|^{-1})\\ &A\to -\mathrm{i}\,\mathrm{grad}(\log\phi)+\mathrm{O}(|z|^{-1})\end{aligned}\tag{4.2}$$

in polar variables $|z|=r\to\infty$ and θ; these vortex boundary conditions justify the use of the bundle ξ with connection A and section ϕ as defined in § 2: (4.2) gives a pedestrian (or physicist's) way of stating the topology of the $\xi=\xi(\beta)$ vector bundle.

The variational problem of (4.1) is to find ϕ and A such that $\delta S/\delta\phi=\delta S/\delta A=0$; if we vary in (4.1) first the section ϕ for a given connection A we obtain

$$\{\bar{D}D+\bar{D}^+D^+\}\phi=\tfrac{1}{2}\gamma\phi(|\phi|^2-1)\tag{4.3a}$$

and likewise varying A for a given section, we get

$$4\partial\bar{\partial}A-4\partial^2A-\mathrm{i}\bar{\phi}\bar{\partial}\phi-2A|\phi|^2=0.\tag{4.3b}$$

If in (4.3) we adopt the boundary conditions (4.2) we 'obtain' the vortex solution of Nielsen–Olesen. In the general case the solution cannot be given in closed form; the original approximate solution (Nielsen and Olesen 1973) has been carefully elaborated by de Vega and Schaposnik (1976) and Jacobs and Rebbi (1979).

Several of the mathematical discussions in §§ 2 and 3 can be read off from (4.3) and its solutions; we hope to have clarified the relationship between the boundary conditions (4.2) and the specific properties of bundles β and $\xi(\beta)$. From the asymptotic solution for the section equation

$$D\phi=0 \qquad (r\to\infty)\tag{4.4}$$

and also from $F(r\to\infty)\to 0$, we immediately obtain the topological quantum number as quantised flux, by applying Stokes' theorem to the field strength (curvature) integral (equations (3.1) and (3.3)).

The angular equation in (4.4) also has quite an interesting physical interpretation for in fact, the relation $r^{-1}\partial_\theta\phi=\mathrm{i}A_\theta\phi$ indicates that the vortex is (at least at large distances) invariant under combined space rotation and internal rotation, because A_θ $(n=1)\sim r^{-1}\partial_\theta\Lambda(\theta)$, where Λ is the map $S^1_\infty\to\mathrm{U}(1)$. This combined rotation invariance is a feature common to many other soliton solutions (Boya *et al* 1978) and in fact is related to the 'spin out of isospin' phenomenon of Hasenfratz and 't Hooft (1976) and others: it turns out that the 'intrinsic' angular momentum is proportional to the topological number.

Also, in the same way that one would ascribe the normal (vector) representation of the SO(2) rotation group ($m=1$) to the section in the tangent bundle to the sphere (Chern class = 2), one reinforces the spinorial character of the vortex because the 'vector' representation is made now in a group which is a *double covering* of the plane rotation group, which would justify an $m=\frac{1}{2}$ representation, viewed from the original group, and in fact if one goes from $r=0$ to $r\to\infty$ in opposite directions, we have $\phi(r\to\infty,\theta)=-\phi(r\to\infty,\pi+\theta)$ as with the kink. But this is a 2π rotation, because the identification $S^2=R^2\cup\{\infty\}$.

One can go further and signal a kind of 'spin statistics theorem': in fact, as shown by Ezawa (1978), the interchange of two vortices with $n=1$ produces a minus sign, i.e. they have the character of fermions.

Finally, we give an application of the complex structure: the topological current

$$j_\mu=\varepsilon_{\mu\alpha\beta}\varepsilon^{ab}\partial^\alpha\phi^a\partial^\beta\phi^b \tag{4.5}$$

is automatically conserved, because of the Kähler condition:

$$\partial^\mu j_\mu=\partial^\mu\varepsilon_{\mu\alpha\beta}\varepsilon^{ab}(\partial^\alpha\phi^a\partial^\beta\phi^b)=\mathrm{d}(\mathrm{d}\phi^a\wedge\mathrm{d}\phi^b)=0 \tag{4.6}$$

which is the $\mathrm{d}w=0$ Kähler condition as before, when passing to the sphere metric. The Kähler form is closed but not exact for the bundle $\xi=\xi(\beta)$

$$\frac{1}{2\pi}\int_{S^2}W=C_1(\xi)=1$$

$$\frac{1}{2\pi}\int_{S^2}W\approx\frac{1}{2\pi}\int_\Sigma \mathrm{d}\alpha=\frac{1}{2\pi}\int_\Sigma\alpha;\qquad \alpha=\mathrm{i}\bar{\phi}\,\bar{\mathrm{d}}\phi/(1+\bar{\phi}\phi)^2; \tag{4.7}$$

it shows how the magnetic flux is 'maintained' by the Higgs field.

5. Morse analysis and stability

We study now the variational problem on the complex vector bundle $\xi=\xi(\beta)$ from the point of view of stability theory. The theory of Morse (Morse 1976, Milnor 1963) relates the topology of the manifold with the indices (= number of negative eigenvalues) of the Hessian (= second derivatives) of a real function on the manifold in the critical (extremal) points. Here our manifold is the set of sections $\Gamma(\xi)$ (over the group of gauge transformations, to be precise), and the function is the energy functional (4.1).

In the set of critical points of the function S we have $\mathrm{d}S=0$ and the Hessian d^2S is well defined as a quadratic form: The Morse index of a solution ϕ is the dimension of the subspace in which the bilinear symmetric form defined by d^2S is negatively defined. We skip technicalities that arise when continuous symmetries (say gauge transformations or translational modes) produce orbits in $\Gamma(\xi)$ (see Bott 1979) and merely state that the number of critical points weighted each one by a factor $(-1)^{\lambda_c}$ (λ_c is the Morse index) is greater than or equal to $C_1(\xi)$; in particular the Morse index has to be even for the vortex bundle.

The vortex will be unstable if its Morse index is different from zero (because $\lambda_c\neq 0$ would provide imaginary frequencies through which the vortex field would 'leak out'). To analyse stability, we have the standard analysis first given by Bogomolny (1976). Let us write S in the form

$$S=\|A\|^2+\|B\|^2+\|F\|^2+\|E\|^2 \tag{5.1}$$

with $A=[D]\phi$, $B=D^+\phi$, $E=2^{-1/2}(|\phi|^2-1)$, and $(\|\cdot\|)$ an L^2 norm. Partial integration and taking the limiting value $\gamma=1$ (analogous to the 'Prasad–Sommerfield limit' in the magnetic monopole soliton of ('t Hooft 1974) yields

$$\|A \pm B\|^2 \mp \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2}\{|AB| + |FE|\} = \|A\|^2 + \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2}|F| \tag{5.2}$$

and then we obtain

$$S = v^2\left\{\int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2}(|D^+\phi|^2 + [F + \tfrac{1}{2}(|\phi|^2 - 1)]^2)\right\} + \left|\int \mathrm{d}z\,\mathrm{d}\bar{z}(\bar{\partial}A - \partial\bar{A})\right|. \tag{5.3}$$

When the equations

$$\begin{aligned} &D^+\phi = 0 \\ &\bar{\partial}A - \partial\bar{A} + \tfrac{1}{2}(|\phi|^2 - 1) = 0 \end{aligned} \tag{5.4}$$

are satisfied, S attains its minimum value, $S = 2\pi v^2 C_1(\xi)$ (Bogomolny 1976). Indeed the vortex solution in the $\gamma = 1$ case satisfies (4.4) (de Vega and Schaposnik 1976), the vortex is stable, and its Morse index is zero; for continuity, $\lambda_c(\text{vortex}) = 0$ for $\gamma \geqslant 1$ also.

6. Index theorem

The index theorem (Atiyah and Singer 1963, Atiyah *et al* 1973) provides, alternatively to Morse theory, a relation between an algebraic index of a differential operator and the differential characterisation of topological invariants. If $\mathscr{D}$ is an *elliptic* (differential) operator, its index is defined by

$$\operatorname{ind}(\mathscr{D}) = \dim\ker\mathscr{D} - \dim\operatorname{coker}\mathscr{D}. \tag{6.1}$$

Here we shall take as $\mathscr{D}$ the first-order differential operator for the *small* variations of equations (5.4). Specifically, if $D = \partial - \mathrm{i}A \to D + \delta D = \partial - \mathrm{i}(A + \delta A)$ and $\phi \to \phi + \delta\phi$ we have

$$\begin{aligned} &D^+\delta\phi = \mathrm{i}\phi\delta\bar{A} \\ &\delta F = \tfrac{1}{2}(\bar{\phi}\delta\phi + \phi\delta\bar{\phi}) \end{aligned} \tag{6.2}$$

or, including the complex conjugate equations and taking the Coulomb gauge $\partial A + \bar{\partial}\bar{A} = 0$, it is $\mathscr{D}\psi = 0$ with

$$\mathscr{D} = \begin{pmatrix} D^+ & 0 & 0 & -\mathrm{i}\phi \\ 0 & \bar{D}^+ & \mathrm{i}\phi & 0 \\ -\frac{1}{2}\phi & \frac{1}{2}\phi & -\bar{\partial} & \partial \\ +\frac{1}{2}\phi & \frac{1}{2}\phi & -\bar{\partial} & \partial \end{pmatrix}, \qquad \psi = \begin{pmatrix} \delta\phi \\ \delta\bar{\phi} \\ \delta A \\ \delta\bar{A} \end{pmatrix}; \tag{6.3}$$

$\mathscr{D}$ can be thought of as an *elliptic* operator. If $\mathscr{D}^* = \text{adjoint}\ \mathscr{D}$ and $\Box_+ = \mathscr{D}^*\mathscr{D}$, $\Box_- = \mathscr{D}\mathscr{D}^*$, we have

$$\operatorname{index}\mathscr{D} = \dim\ker\Box_+ - \dim\ker\Box_-. \tag{6.4}$$

In the 'heat equation' approach to the index theorem (Atiyah *et al* 1973) we construct the function

$$h_t(\Box_+) = \sum_r \exp(-\lambda_r t)\dim\Gamma_{\lambda_r} = n_+ + \sum_r{}' \exp(-\lambda_r t)\dim\Gamma_{\lambda_r} \tag{6.5}$$

with Γ_{λ_r} subspace of λ_r eigenmodes, $n_+ = \dim \Gamma_{\lambda_0}$, $\Sigma_r' = \Sigma_{r\neq 0}$. Then

$$\operatorname{ind}(\mathscr{D}) = h_t(\Box_+) - h_t(\Box_-) = n_+ - n_-$$

because $\Box_+$ and $\Box_-$ differ only on the null space.

In order to identify ind($\mathscr{D}$) with the topological index, we observe that formally

$$h_t(\Box_+) = \operatorname{Tr}\exp(-\Box_+ t). \tag{6.6}$$

Therefore, if $\Box_+ = \Sigma_r \lambda_r P_r$ is the eigenvalue expansion we have

$$\begin{aligned} h_t(\Box_+) &= \operatorname{Tr}\exp\Big(-\sum_r \lambda_r P_r t\Big) = \operatorname{Tr}\Big(\sum_r P_r \exp(-\lambda_r t)\Big) \\ &= \lim_{z'\to z} \sum_r \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2} \langle z|r\rangle \exp(-\lambda_r t)\langle r|z'\rangle \\ &= \lim_{z'\to z} \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2} \Big(\sum_r \psi_r(z)\psi_r^*(z')\exp(-\lambda_r t)\Big). \end{aligned} \tag{6.7}$$

The most singular part of (6.7) occurs for $t \to 0$; namely, this part is dominated by the asymptotic expansion, in the sense of the Wilson operator product expansion, by the dim = 1 gauge-invariant term F, according to equation (4.23) of Atiyah *et al* (1973). Therefore

$$h_t^{\mathrm{reg}}(\Box_+) = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{4\pi}(\bar{\partial}A - \partial\bar{A}) + \int \mathrm{d}z\,\mathrm{d}\bar{z}\Big(\sum_r{}' \exp(-\lambda_r t)\psi_r^*(z)\psi_r(z)\Big) \tag{6.8}$$

and

$$\operatorname{ind}(\mathscr{D}) = h_t^{\mathrm{reg}}(\Box_+) - h_t^{\mathrm{reg}}(\Box_-) = \frac{1}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,F = 2nC_1(\xi).$$

There exist $2n$ modes of fluctuations of zero frequency, corresponding to the n complex position coordinates of the n vortices. This result implies stability in the $\gamma = 1$ limit, even if we do not know exactly the n-vortex solutions with their centres arbitrarily located (Weinberg 1979). The vortices do not interact at the critical $\gamma = 1$ value, the transition point between type-II and type-I superconductivity.

7. Fermions and spin bundles

The spinor and fermion nature of the vortex illustrates also the recent work of Jackiw and Schrieffer (1981) on fractional charged fermions. In the spinor bundle

$$\begin{array}{l} \quad\;\; \mathrm{Spin}\,(2) \\ \quad\;\;\; \downarrow \\ Y = C^2 \to E_S \to S^2 \end{array}$$

the vector space Y supports the irreducible complex representation of dimension 2 of Spin (2) = $\widetilde{\mathrm{SO}}(2) \approx \mathrm{SO}(2)$. A Dirac operator $\not{D}_v = \Gamma(E_S) \to \Gamma(E_S)$ may be defined in it (Palais 1965). This is an elliptic operator, whose index (see § 6) is obtained (for example) by a spectral equation which is the Dirac equation in an external field, namely the connection provided by the vector potential of the vortex (in other words, we 'pass' the connection, which is originally given in the principal bundle $\beta: S^1 \to S^3 \to S^2$, first to the associated bundle of the vortex $S^1 \hookrightarrow C$, and then to the associated bundle of the spinors, $S^1 = \mathrm{Spin}(2) \hookrightarrow \mathrm{C}^2$).

It is fairly well known how the equation

$$\not{D}_v \psi_\lambda = \lambda \psi_\lambda \tag{7.1}$$

gives index $\not{D} = 1$ by studing the zero modes (Nielsen and Schroer 1977), $\lambda = 0$. To see what the electric charge of the fermion ground state is, we proceed as follows: the electric charge of the ground state (Dirac sea) is the difference between the charge of negative-energy states in the vortex background field ψ_λ^v and the zero field ψ_λ^0:

$$Q = \int \frac{dz\, d\bar{z}}{2} \int_{-\infty}^{0} d\lambda \{\rho_\lambda^v(z) - \rho_\lambda^0(z)\}$$

with

$$\rho_\lambda^v(z) = \bar{\psi}_\lambda^v(z)\psi_\lambda^v(z) \tag{7.2}$$

and ρ_λ^0 the spectral density for the free Dirac equation. The discrete chiral transformation anticommutes with the energy, $\gamma_5\psi_\lambda(z) = \psi_{-\lambda}(z)$, $\rho_\lambda^v \to \rho_{-\lambda}^v$. Subtracting the completeness relation $\int_{-\infty}^{\infty} d\lambda\, \psi_\lambda(z)\bar{\psi}_\lambda(z') = \delta(z - z')$ for the two cases we obtain

$$\begin{aligned} Q &= \int_{-\infty}^{\infty} d\lambda [\rho_\lambda^v(z) - \rho_\lambda^0(z)] \\ &= \int_{-\infty}^{0} d\lambda [2\rho_\lambda^v(z) + \bar{\psi}_0^v(z)\psi_0^v(z) - 2\rho_\lambda^0(z)] \end{aligned} \tag{7.3}$$

and only the zero eigenvalue contributes; as they can be computed from the index theorem (e.g. by the heat equation kernel), we finally get (in this 'two-dimensional' fibre bundle):

$$\begin{aligned} Q &= -\frac{1}{2}\int \frac{dz\, d\bar{z}}{2} \bar{\psi}_0^v(z)\psi_0^v(z) = -\tfrac{1}{2}\{h_t(\not{D}_v^+\not{D}_v) - h_t(\not{D}_v\not{D}_v^+)\} \\ &= -\tfrac{1}{2} \end{aligned} \tag{7.4}$$

thus obtaining that the 'vacuum' state in the background field of a vortex electromagnetic field has a *half-integer* charge: the topological structure produces *vacuum polarisation*: part of the charge 'leaks out' to the background.

This can be understood perhaps in the light of the considerations of § 4: namely that the balancing of angular momentum and charge ('isospin' where the group is SO(2) instead of SO(3)) is $(\frac{1}{2}, \frac{1}{2})$, as the vortex configuration itself has invariance under combined rotations only; the 'left-over' charge $-\frac{1}{2}$ is just in balance with the $\frac{1}{2}$ value of the vector representation of the group SO(2) of the vortex, seen from the ordinary SO(2) group. In fact, this balancing is ostensible in the very first discussion of the fermion charge fractionation by Jackiw and Schrieffer (1981).

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